

Stabilization with target oriented control for higher order difference equations

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Abstract

For a physical or biological model whose dynamics is described by a higher order difference equation $u_{n+1} = f(u_n, u_{n-1}, \dots, u_{n-k+1})$, we propose a version of a target oriented control $u_{n+1} = cT + (1 - c)f(u_n, u_{n-1}, \dots, u_{n-k+1})$, with $T \geq 0$, $c \in [0, 1]$. In ecological systems, the method incorporates harvesting and recruitment and for a wide class of f , allows to stabilize (locally or globally) a fixed point of f . If a point which is not a fixed point of f has to be stabilized, the target oriented control is an appropriate method for achieving this goal. As a particular case, we consider pest control applied to pest populations with delayed density-dependence. This corresponds to a proportional feedback method, which includes harvesting only, for higher order equations.

Keywords: Chaos, target oriented control, higher order difference equation, globally asymptotically stable fixed point, delay Ricker model, Pielou equation

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1. Introduction

Controlling chaos consists in stabilizing nonlinear systems with chaotic dynamics [1, 8, 24]. The target state for control can be a steady state, a periodic orbit, or even a particular aperiodic trajectory in a chaotic attractor. The best way to achieve any of these goals depends on the nature of the situation modeled by the system. For instance, in population dynamics, control methods incorporating either harvesting or stocking, or both, are more appropriate than others, e.g. OGY method [23], which performs continuous small perturbations in the parameters of the system.

Using OGY method, stabilization of otherwise unstable fixed point or cycle was achieved experimentally in [11]. The experimental system consisted of a gravitationally buckled, amorphous magnetoelastic ribbon. The chosen ribbon material exhibited very large reversible changes of Young's modulus with the application of small magnetic fields. Oscillation of the ribbon were brought to a chosen regime using the control of the distance from the chosen orbit [11]. However, the stabilized orbit had to be an orbit of the unperturbed system.

The main motivation of our investigation is population dynamics where chaotic orbits coming close enough to zero can threaten population survival and cause unpredictable changes in a relevant food chain. Higher density values may be as dangerous as low ones due to either extinction stipulated by the previous overpopulation and accumulated pollution [26], or connected to paradox of enrichment [27]. If the target is pest eradication then harvesting at each time step can be an appropriate strategy. However, if the purpose is to keep the population in certain bounds, the control should incorporate both harvesting and stock recruitment. A recently suggested target oriented control [9] seems to be an adequate method to achieve this goal: the control intensity, as in OGY method, depends on the deviation of the stock size from the chosen target—the more distant the population density from the target value, the more intensive the control is. Since 2011, when the target oriented control was introduced, there were several developments [5, 13] justifying a possibility to stabilize a fixed point (for a prescribed target) and exploring the form of the stock-recruitment dependency when such stabilization is possible. The purpose of the present paper is to overcome the following shortcomings of previous contributions:

1. Most of control methods considered suitable for stabilizing population dynamics have been studied in the framework of first order difference equations [6, 9, 12, 15, 16, 18, 21, 28, 29]. In any physical application, this means that we consider a scalar system only, and this system has no memory: the next state depends on the previous state only. In biological applications, first order equations can describe non-structured populations that survive for one season only, with overwintering offspring. The framework of higher order equations allows to consider multi-seasonal interactions even with non-overlapping generations. Moreover, dynamics of a structured population in certain cases can be described by a higher order difference equation; e.g. [14].
2. OGY and several similar methods (for example, prediction based control [6]) are focused on stabilization of either a fixed point or an unstable orbit of the original system. Other strategies, such as proportional feedback [16], involve harvesting only and stabilize a point which is typically closer to zero than fixed points of the non-controlled equation. However, physical or ecological considerations (for example, required density to sustain functioning of a food chain or providing a sufficient supply for harvesting) lead to the necessity to stabilize a point which *is not* a fixed point of the system. Here we illustrate that, with target oriented control, it is possible. It will allow to keep a controlled population at a prescribed level, certainly, at a cost.

The present paper is devoted to a general control strategy with the goal of stabilizing steady states of a broad class of higher order difference equations which include chaotic and non-chaotic systems.

We recall that first order difference equations

$$u_{n+1} = h(u_n), \quad n = 0, 1, 2, \dots \quad (1.1)$$

are suitable for modeling single species populations with non-overlapping generations. The map h usually takes the form $h(x) = xg(x)$ with g being a positive decreasing map regulating

the intraspecific competition for resources. The Ricker model [26], $u_{n+1} = u_n \exp(r - u_n)$, and the Beverton-Holt model [4], $u_{n+1} = ru_n/(1 + u_n)$, are examples of (1.1) broadly used in theoretical ecology. However, the evidence of the existence of explicit time lags in the intraspecific regulatory mechanisms for some species [20] makes it necessary to incorporate delays in the equations to obtain more realistic models. Examples of such equations related to the Ricker and Beverton-Holt models are the delay Ricker equation [20]

$$u_{n+1} = u_n \exp(r - u_{n-k+1}) \quad (1.2)$$

and the Pielou equation [25]

$$u_{n+1} = \frac{ru_n}{1 + u_{n-k+1}}, \quad (1.3)$$

where $k \geq 2$ is a fixed natural number determining the time lag. Clearly, both (1.2) and (1.3) are special cases of the k th-order difference equation

$$u_{n+1} = f(u_n, u_{n-1}, \dots, u_{n-k+1}), \quad n = 0, 1, 2, \dots \quad (1.4)$$

where $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty)$. Moreover, similarly to the first order equation (1.1), the higher order difference equation (1.4), and particularly (1.2) and (1.3), can show complicated dynamics. Therefore, the design and study of control strategies for the higher order equation (1.4) is a natural extension of the stabilization problem for the first order equation (1.1).

Here, we generalize a method called target oriented control. Target oriented control (TOC) method

$$u_{n+1} = h(cT + (1 - c)u_n), \quad T \geq 0, \quad c \in [0, 1), \quad (1.5)$$

was developed in [9] with the aim to stabilize the dynamics of the first order equation (1.1). Parameter T is called target and parameter c measures the control intensity. Essentially, TOC increases the state variable if it is smaller than the target and reduces it if it is larger. For the values of c close enough to one, TOC can provide global stabilization of unimodal maps with a negative Schwarzian derivative [13] and some other models, where the smoothness conditions are relaxed [5]. Following [13], we note that (1.5) is a combination of the linear transformation of the variable

$$\phi(x) = cT + (1 - c)x \quad (1.6)$$

and the function h . Moreover, if we switch the order and consider the modified target oriented control (MTOC)

$$u_{n+1} = cT + (1 - c)h(u_n), \quad T \geq 0, \quad c \in [0, 1), \quad (1.7)$$

then the fixed point K_c of the controlled equation (1.5) is globally (locally) asymptotically stable if and only if the fixed point $P_c = \phi(K_c)$ of (1.7) is globally (locally) asymptotically stable.

The framework of MTOC provides a natural generalization of target control methods to higher order difference equations. In particular, we propose the following control applied to the uncontrolled equation (1.4)

$$u_{n+1} = cT + (1 - c)f(u_n, u_{n-1}, \dots, u_{n-k+1}), \quad T \geq 0, \quad c \in [0, 1). \quad (1.8)$$

Further we will refer to the controlled equation (1.8) as HMTOC (higher-order modified target oriented control).

If we assume the zero target $T = 0$ in (1.5), then we obtain the proportional feedback method (PF),

$$u_{n+1} = h((1 - c)u_n), \quad c \in [0, 1), \quad (1.9)$$

consisting in a reduction of the state variable, proportional to the size of this variable [16]. The assumption of the proportional reduction is aligned with the idea of constant effort harvesting, without any stocking. In (1.9), harvesting occurs before reproduction. Switching the variable reduction function $\psi(x) = (1 - c)x$ with the map h , we get a modified proportional feedback method (MPF) in which harvesting takes place after reproduction

$$u_{n+1} = (1 - c)h(u_n), \quad c \in [0, 1). \quad (1.10)$$

Similarly, the control in (1.10) can be extended to involve higher order methods. We obtain a modified version of the proportional feedback control for higher order equations

$$u_{n+1} = (1 - c)f(u_n, u_{n-1}, \dots, u_{n-k+1}), \quad c \in [0, 1), \quad (1.11)$$

which is a particular case of HMTOC for the target $T = 0$.

The main results of the present paper are the following:

1. We obtain sufficient conditions for local and global stabilization of a fixed point with HMTOC. As illustrated by numerical examples, some of these conditions are sharp.
2. Stabilization of the zero equilibrium with the proportional feedback method is considered as a particular case of the above results.
3. The minimal stabilizing control intensity c is estimated. The flexibility for the choice of the point to be stabilized is one of the advantages of HMTOC. However, stabilization conditions depend on this choice.

The paper is organized as follows. Section 2 deals with local stabilization of higher order equations. In Section 3, we justify the possibility of global stabilization. Section 4 contains the proof of the fact that with target oriented control, any prescribed point can be stabilized. Some examples and numerical illustrations are presented in Section 5. Finally, Section 6 contains a summary.

2. Local stabilization

Our first result gives a sufficient condition for HMTOC to have at least a positive fixed point for all control intensities. It complements and slightly generalizes Lemma 1 in [13]. All the proofs of results in this section can be found in the Appendix.

Lemma 2.1. *(i) Assume that $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ is continuous and there exists a positive constant M such that $f(M, \dots, M) \leq M$. Then, for any $T \in (0, M]$ there exists at least a positive fixed point P_c of HMTOC in $(0, M]$ for every $c \in (0, 1)$.*

(ii) In particular, if there exists a positive constant \overline{M} such that $f(x, \dots, x) \leq x$ for $x \geq \overline{M}$, then for any fixed target $T > 0$ there exists at least a positive fixed point P_c of HMTOC in the interval $(0, \max\{T, \overline{M}\}]$ for every $c \in (0, 1)$.

Next, we show that HMTOC is able to asymptotically stabilize a fixed point if a sufficiently strong control is implemented.

Theorem 2.2. *Assume that $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ is continuously differentiable and that there exists a bounded interval $I \subset \mathbb{R}_+$ such that HMTOC with target $T > 0$ has at least a fixed point P_c in I for every $c \in (0, 1)$. Then, there exists $c^* \geq 0$ such that P_c is asymptotically stable for $c \in (c^*, 1)$.*

From a practical perspective, it is convenient to have an estimation as good as possible for c^* in the statement of Theorem 2.2. Following its proof, this essentially reduces to prove that a polynomial is a Schur polynomial (i.e. with all roots smaller than one in module). Although there are characterizations for a Schur polynomial, as for example the Schur test or the Jury conditions [17, Corollaries 3.4.91 and 3.4.98], obtaining an explicit expression for c^* is difficult for two main reasons. First, because if k is large, then expressions in these characterizations get too complicated. Second, because we need to know with precision the interval I where the fixed point P_c of HMTOC is, in order to reduce the range of possible values for the partial derivatives of f . Of course, the second issue disappears if the target $T = K$ is a fixed point of f , that is,

$$K = f(K, K, \dots, K), \quad (2.1)$$

because then K is also a fixed point of HMTOC for every $c \in (0, 1)$, and the interval I reduces to the point K . In the next result, we take advantage of this fact and present a sharp estimation of the control intensity necessary to stabilize a fixed point K of f when $k = 2$ and the target coincides with the fixed point $T = K$.

Theorem 2.3. *Assume that $f: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is continuously differentiable and $K \in \mathbb{R}_+$ satisfies $K = f(K, K)$. Then, the fixed point K of the controlled equation HMTOC with the target $T = K$ is asymptotically stable for $c \in (c^*, 1)$ where*

$$c^* = \max \left\{ 0, 1 - \frac{1}{\max\{|\frac{\partial f}{\partial y}(\mathbf{K})|, |\frac{\partial f}{\partial x}(\mathbf{K})| + \frac{\partial f}{\partial y}(\mathbf{K})\}} \right\},$$

($c^* = 0$ if $\frac{\partial f}{\partial x}(\mathbf{K}) = \frac{\partial f}{\partial y}(\mathbf{K}) = 0$), and $\mathbf{K} = (K, K)$.

As we have said, if k is larger, then optimal expressions for c^* get complicated. Nevertheless, when (2.1) holds, it is possible to get easy-to-calculate estimates such as the following one.

Theorem 2.4. *Assume that $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ is continuously differentiable and $K \in \mathbb{R}_+$ satisfies $K = f(K, K, \dots, K)$. Then, the fixed point K of the controlled equation HMTOC with the target $T = K$ is asymptotically stable for $c \in (c^*, 1)$, where*

$$c^* = \max \left\{ 0, 1 - \frac{1}{\sum_{j=1}^k |\frac{\partial f}{\partial x_j}(\mathbf{K})|} \right\},$$

$\mathbf{K} = (K, K, \dots, K)$.

3. Global stabilization of a fixed point

In this section, we assume that the point $K \in \mathbb{R}_+$ to be stabilized is a fixed point for the uncontrolled system, that is, it satisfies (2.1). Further, we assume that there exists $L > 0$ such that

$$|f(\mathbf{x}) - K| \leq L \|\mathbf{x} - (K, K, \dots, K)\|, \quad \mathbf{x} \in \mathbb{R}_+^k \quad (3.1)$$

where $\mathbf{x} = (x_1, \dots, x_k)$ and $\|\mathbf{x}\| = \max_{1 \leq j \leq k} |x_j|$, however the proofs below are easily adapted to any other vector norm. We note that inequality (3.1) implies that K is a fixed point of f .

Our first result gives not only a sufficient condition for global stability of a fixed point of the controlled equation but also an estimate of the control intensity necessary to reach it, which depends on the constant L in condition (3.1). Its proof uses some ideas from [3].

Theorem 3.1. *Assume that $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ satisfies condition (3.1). Then for $c \in (c^*, 1)$ with $c^* = \max\{0, 1 - \frac{1}{L}\}$, the fixed point K is a globally asymptotically stable fixed point for the controlled equation HMTOC with $T = K$, that is, any sequence starting with $(u_{1-k}, u_{2-k}, \dots, u_0) \in \mathbb{R}_+^k$ and satisfying HMTOC with $T = K$ converges to K .*

Proof. Without loss of generality, let us assume that $L \geq 1$ in inequality (3.1). We fix $\theta \in (0, 1)$ and let $c = 1 - \frac{\theta}{L}$, where $c \in (0, 1)$. Then, using equation (1.8) and inequality (3.1), we have

$$\begin{aligned} |u_{n+1} - K| &= (1 - c)|f(u_n, u_{n-1}, \dots, u_{n-k+1}) - K| \\ &= \frac{\theta}{L}|f(u_n, u_{n-1}, \dots, u_{n-k+1}) - K| \\ &\leq \theta \frac{L}{L} \|(u_n - K, u_{n-1} - K, \dots, u_{n-k+1} - K)\| \\ &= \theta \max_{n-k+1 \leq j \leq n} |u_j - K|, \end{aligned}$$

therefore

$$|u_{n+1} - K| \leq \theta \max_{n-k+1 \leq j \leq n} |u_j - K|.$$

Continuing this process, we obtain

$$|u_{n+l} - K| \leq \theta \max_{n-k+1 \leq j \leq n} |u_j - K|, \quad l = 1, \dots, k. \quad (3.2)$$

Using (3.2) as an induction step, we obtain

$$|u_n - K| \leq \theta^{[n/k]} \max_{-k+1 \leq j \leq 0} |u_j - K|,$$

where $[t]$ is the integer part of t . Then

$$\lim_{n \rightarrow \infty} u_n = K,$$

moreover, there is a guaranteed rate of convergence. Thus, K is a globally asymptotically stable fixed point of equation (1.8) for any $c \in (c^*, 1)$ with

$$c^* = 1 - \frac{1}{L}.$$

□

Applying Theorem 3.1 to (1.11), we obtain the following result about the stabilization of the origin with proportional control.

Corollary 3.2. *Assume that $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ satisfies condition (3.1) with $K = 0$. Then for $c \in (c^*, 1)$ with $c^* = \max\{0, 1 - \frac{1}{L}\}$ zero is a globally asymptotically stable fixed point for the controlled equation (1.11).*

In Theorem 3.1 and Corollary 3.2, constant L in (3.1) is used to estimate the control intensity necessary to stabilize globally a fixed point: the smaller the value of L the sooner the global stability is attained. In some cases the calculation of L can be direct, as for the Pielou equation with $r \geq 1$ where

$$f(\mathbf{x}) = f(x_1, \dots, x_k) = \frac{rx_1}{1 + x_k}$$

satisfies (3.1) with $K = r - 1$ and $L = r$:

$$\begin{aligned} |f(\mathbf{x}) - K| &= \left| \frac{rx_1}{1 + x_k} - (r - 1) \right| = \left| \frac{rx_1 - (r - 1)x_k - (r - 1)}{1 + x_k} \right| \\ &= \left| \frac{r(x_1 - (r - 1))}{1 + x_k} - \frac{(r - 1)(x_k - (r - 1))}{1 + x_k} \right| \\ &\leq r|x_1 - (r - 1)| + (r - 1)|x_k - (r - 1)| \leq L\|\mathbf{x} - (K, \dots, K)\|, \quad \mathbf{x} \in \mathbb{R}_+^k. \end{aligned}$$

However, in general finding L could be hard. Therefore, it is interesting to have easy ways to calculate L for a given map. Evidently, if f is globally Lipschitz continuous and K is a fixed point of f , then we could take L as the global Lipschitz constant of f . The proof of the next result shows how to calculate L if f is a locally Lipschitz continuous bounded function.

Lemma 3.3. *Let $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ be a locally Lipschitz continuous bounded function and K be a fixed point of f , then there exists $L \geq 1$ such that condition (3.1) holds for $\mathbf{x} \in \mathbb{R}_+^k$.*

Proof. Since f is bounded, there exists $A > 0$ such that

$$0 \leq f(\mathbf{x}) \leq A, \quad \mathbf{x} \in \mathbb{R}_+^k. \quad (3.3)$$

Moreover, the local Lipschitz continuity of f guarantees that condition (3.1) is satisfied for $\mathbf{x} = (x_1, x_2, \dots, x_k)$ such that $0 \leq x_i \leq 2K$, $i = 1, 2, \dots, k$ with some constant \tilde{L} .

Next, let at least one of x_i satisfy $x_i > 2K$, then

$$\|\mathbf{x} - (K, K, \dots, K)\| > K. \quad (3.4)$$

Due to (3.3) and $A \geq K > 0$, we have

$$|f(\mathbf{x}) - K| \leq \max\{A - K, K\},$$

which together with (3.4) gives

$$|f(\mathbf{x}) - K| \leq \max\left\{\frac{A}{K} - 1, 1\right\} \|\mathbf{x} - (K, K, \dots, K)\| \quad (3.5)$$

for $\mathbf{x} \notin [0, 2K]^k$. Thus, we obtain that

$$|f(\mathbf{x}) - K| \leq L \|\mathbf{x} - (K, K, \dots, K)\|, \quad \mathbf{x} \in \mathbb{R}_+^k$$

holds with $L = \max\left\{\tilde{L}, \frac{A}{K} - 1, 1\right\}$. □

4. Stabilization of an arbitrary point

The results contained in previous section allow us to consider stabilization of a fixed point only. If we aim to stabilize a different point with HMTOC, then we need to apply a “corrective” HMTOC first. The next result shows that after applying both such a corrective step and a stabilizing control, we are still using HMTOC.

Lemma 4.1. *A combination of two HMTOCs is a HMTOC.*

Proof. If $\phi_1(x) = c_1 T_1 + (1 - c_1)x$ is applied after another argument transformation $\phi_2(x) = c_2 T_2 + (1 - c_2)x$, then

$$\begin{aligned} \phi_1(\phi_2(f(x_1, x_2, \dots, x_k))) &= c_1 T_1 + (1 - c_1)[c_2 T_2 + (1 - c_2)f(x_1, x_2, \dots, x_k)] \\ &= c_1 T_1 + (1 - c_1)c_2 T_2 + (1 - c_1)(1 - c_2)f(x_1, x_2, \dots, x_k) \\ &= c_3 T_3 + (1 - c_3)f(x_1, x_2, \dots, x_k), \end{aligned}$$

where

$$c_3 = c_1 + c_2 - c_1 c_2, \quad \text{and} \quad T_3 = \frac{c_1 T_1 + c_2 T_2 - c_1 c_2 T_2}{c_3}.$$

Obviously $\alpha := (1 - c_1)(1 - c_2) \in (0, 1)$ as long as $c_1, c_2 \in (0, 1)$, so $c_3 = 1 - \alpha \in (0, 1)$. On the other hand, the value of T_3 is positive as a ratio of two positive numbers. □

The following result was justified in [5, Lemma 5].

Lemma 4.2. *Let $f_1: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function satisfying $f_1(x) > 0$ for $x > 0$. Then for any $K > 0$ in the range of f there exist $c_K \in (0, 1)$ and $T_K \geq 0$ such that K is a fixed point of $g(x) = c_K T_K + (1 - c_K)f_1(x)$.*

Lemma 4.2 immediately implies the following result.

Corollary 4.3. *Let $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ be a continuous function satisfying $f(x, x, \dots, x) > 0$ for $x > 0$. Then for any $K \in \{f(x, x, \dots, x) : x > 0\}$ there exist $c_K \in (0, 1)$ and $T_K \geq 0$ such that K is a fixed point of $g(\mathbf{x}) = c_K T_K + (1 - c_K)f(\mathbf{x})$.*

Proof. Apply Lemma 4.2 with $f_1(x) = f(x, x, \dots, x)$. \square

Let us demonstrate that for suitable functions any point can be stabilized with a combination of HMTOCs.

Theorem 4.4. *Let $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ be a continuous differentiable function with $f(x, x, \dots, x) > 0$ for $x > 0$. Then for any $K_1 \in \{f(x, x, \dots, x) : x > 0\}$ there exists a combination of two HMTOCs for which K_1 is an asymptotically stable fixed point.*

Additionally, let either $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ be globally Lipschitz continuous, or let (3.1) hold for a fixed point K , or f be a globally bounded function. Then for any $K_1 \in \{f(x, x, \dots, x) : x > 0\}$ there exists a combination of two HMTOCs for which K_1 is a globally asymptotically stable fixed point.

Proof. By Corollary 4.3 there exists $c_{K_1} \in (0, 1)$ and $T_{K_1} \geq 0$ such that K_1 is a fixed point of $g(\mathbf{x}) = c_{K_1} T_{K_1} + (1 - c_{K_1})f(\mathbf{x})$.

Therefore, we can apply Theorem 2.2 to guarantee that there exists $c^* \in (0, 1)$ such that for $c \in (c^*, 1)$ the combination of HMTOCs

$$u_{n+1} = cK_1 + (1 - c)[c_{K_1} T_{K_1} + (1 - c_{K_1})f(u_n, u_{n-1}, \dots, u_{n-k+1})] \quad (4.1)$$

has K_1 as an asymptotically stable fixed point.

Additionally, let either $f: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ be globally Lipschitz continuous, or condition (3.1) hold for a fixed point K , or f be a globally bounded function. We note that the conditions of the theorem imply that f is locally Lipschitz. From now on, we write $\mathbf{K} = (K, K, \dots, K)$ and $\mathbf{K}_1 = (K_1, K_1, \dots, K_1)$. Let us assume that after the first HMTOC, the function $g(\mathbf{x}) = c_{K_1} T_{K_1} + (1 - c_{K_1})f(\mathbf{x})$ satisfies $g(\mathbf{K}_1) = K_1$, therefore $K_1 - c_{K_1} T_{K_1} = (1 - c_{K_1})f(\mathbf{K}_1)$. Thus

$$\begin{aligned} |g(\mathbf{x}) - K_1| &= |c_{K_1} T_{K_1} + (1 - c_{K_1})f(\mathbf{x}) - K_1| \\ &= |(1 - c_{K_1})[f(\mathbf{x}) - f(\mathbf{K}_1)]| \\ &\leq L(1 - c_{K_1})\|\mathbf{x} - \mathbf{K}_1\|, \quad \mathbf{x} \in \mathbb{R}_+^k, \end{aligned}$$

if f is globally Lipschitz with the constant L . In consequence, inequality (3.1) holds for g and K_1 .

Next, consider the case when f satisfies (3.1) for a fixed point K . By the local Lipschitz condition, it is possible to choose $L_2 > 0$ such that

$$\begin{aligned} |g(\mathbf{x}) - K_1| &= (1 - c_{K_1})|f(\mathbf{x}) - f(\mathbf{K}_1)| \\ &\leq (1 - c_{K_1})L_2\|\mathbf{x} - \mathbf{K}_1\| \quad \text{for} \quad \|\mathbf{x} - \mathbf{K}_1\| \leq \max\{K, K_1\}, \quad \mathbf{x} \in \mathbb{R}_+^k. \end{aligned} \quad (4.2)$$

Moreover, for any $\mathbf{x} \in \mathbb{R}_+^k$ such that $\|\mathbf{x} - \mathbf{K}_1\| \geq \max\{K, K_1\}$, we have $\|\mathbf{x} - \mathbf{K}_1\| \geq K_1 = \|\mathbf{K}_1\|$ and $\|\mathbf{x} - \mathbf{K}_1\| \geq K = \|\mathbf{K}\|$. Hence

$$\begin{aligned}
|g(\mathbf{x}) - K_1| &= (1 - c_{K_1}) |f(\mathbf{x}) - f(\mathbf{K}_1)| \\
&\leq (1 - c_{K_1}) |f(\mathbf{x}) - K| + (1 - c_{K_1}) |f(\mathbf{K}_1) - K| \\
&\leq L(1 - c_{K_1}) \|\mathbf{x} - \mathbf{K}\| + L(1 - c_{K_1}) \|\mathbf{K}_1 - \mathbf{K}\| \\
&= L(1 - c_{K_1}) \|\mathbf{x} - \mathbf{K}_1 + \mathbf{K}_1 - \mathbf{K}\| + L(1 - c_{K_1}) [\|\mathbf{K}\| + \|\mathbf{K}_1\|] \\
&\leq L(1 - c_{K_1}) \|\mathbf{x} - \mathbf{K}_1\| + 2L(1 - c_{K_1}) [\|\mathbf{K}\| + \|\mathbf{K}_1\|] \\
&\leq L(1 - c_{K_1}) \|\mathbf{x} - \mathbf{K}_1\| + 4L(1 - c_{K_1}) \|\mathbf{x} - \mathbf{K}_1\|.
\end{aligned}$$

Thus

$$|g(\mathbf{x}) - K_1| \leq L_1 \|\mathbf{x} - \mathbf{K}_1\|, \quad \mathbf{x} \in \mathbb{R}_+^k$$

where $L_1 = (1 - c_{K_1}) \max\{L_2, 5L\}$, and L_2, L appear in (4.2) and (3.1), respectively.

Finally, a similar argument proves that inequality (3.1) holds for g and K_1 when f is globally bounded.

Further, Theorem 3.1 guarantees that there exists $c^* \in (0, 1)$ such that for $c \in (c^*, 1)$ the combination of HMTOCs has K_1 as a globally asymptotically stable fixed point. \square

5. Examples and numerical simulations

In this section, three examples illustrate how the previous results can be used to determine a control intensity sufficient to stabilize a steady state.

5.1. Stabilization of a nontrivial fixed point

Let us consider the second order equation

$$u_{n+1} = \exp(1 - u_n) \exp(1 - u_{n-1}^2), \quad (5.1)$$

which has a stable 3-cycle and an unstable fixed point $K = 1$. Assume that we are interested in stabilizing such a fixed point. To estimate the control intensity for that goal, we begin by calculating the partial derivatives of the map $f(x, y) = \exp(1 - x) \exp(1 - y^2)$,

$$\frac{\partial f}{\partial x}(x, y) = -\exp(1 - x) \exp(1 - y^2), \quad \frac{\partial f}{\partial y}(x, y) = -2y \exp(1 - x) \exp(1 - y^2).$$

Evaluating them at $\mathbf{K} = (1, 1)$ and using Theorem 2.3, $K = 1$ is a locally stable fixed point for the controlled equation HMTOC with the target $T = 1$ if $c \in (c^*, 1)$ with

$$c^* = 1 - \frac{1}{\max\{|-2|, |-1| - 2\}} = 1 - \frac{1}{2} = 0.5.$$

Moreover, the partial derivatives of f satisfy

$$\left| \frac{\partial f}{\partial x}(x, y) \right| = |-\exp(1 - x) \exp(1 - y^2)| \leq e^2 \approx 7.39, \quad (x, y) \in \mathbb{R}_+^2,$$

and

$$\left| \frac{\partial f}{\partial y}(x, y) \right| = | -2y \exp(1-x) \exp(1-y^2) | \leq 2e \frac{\sqrt{e}}{\sqrt{2}} \approx 6.34, \quad (x, y) \in \mathbb{R}_+^2,$$

where we have used that, in \mathbb{R}_+ , the function $\exp(1-x)$ is decreasing and the function $x \exp(1-x^2)$ is bounded by $\sqrt{e}/\sqrt{2}$.

In consequence, by the mean value theorem in several variables and the Cauchy-Schwarz inequality, map f is globally Lipschitz continuous with constant $L = 2e^2 \approx 14.78$ and condition (3.1) holds for the same L . Using Theorem 3.1 we obtain that the fixed point $K = 1$ is a globally asymptotically stable fixed point of the controlled equation HMTOC for at least any c greater than $1 - \frac{1}{2e^2} \approx 0.93$.

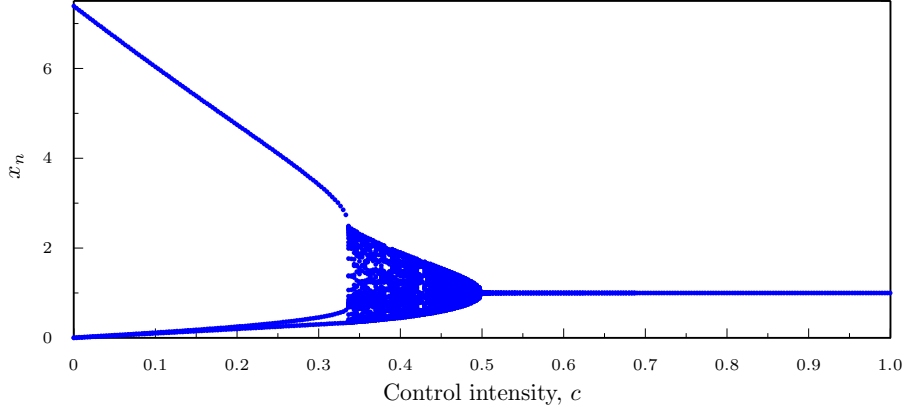


Figure 1: Stabilizing effect of HMTOC on the uncontrolled equation $u_{n+1} = \exp(1 - u_n) \exp(1 - u_{n-1}^2)$. Target was chosen as $T = 1$, which is a fixed point of the uncontrolled system. For each $c \in k/300$, $k = 1, \dots, 300$, we plotted 50 consecutive values of u_n after discarding the first 3000. Initial conditions were chosen pseudo-randomly.

Figure 1 shows the effect of increasing the control intensity c from 0 to 1 in equation (5.1). We observe that the interval of control intensities guaranteeing local stability of the fixed point is sharp. However, numerically it seems that the interval of control intensities guaranteeing the global attraction derived from Theorem 3.1 could be improved. We emphasize that such an improvement could be obtained directly from Theorem 3.1, if we show that condition (3.1) holds for a smaller L . Next example illustrates that using the best constant L in condition (3.1) can give sharp results for global attraction.

5.2. Pest eradication. Stabilization of the trivial fixed point.

Let us consider the delayed Ricker equation $u_{n+1} = u_n \exp(1.5 - u_{n-1})$, which has been proposed as a model for populations with non-overlapping generations and a one generation time lag in the intraspecific competition for resources [20]. Suppose that we want to stabilize the fixed point $K = 0$. For example, this could be the case if the population is a pest and we are interested in eradicating it. We choose the target $T = 0$ in HMTOC, which biologically corresponds to harvesting a constant proportion of the population after reproduction.

Then, analogously to the previous example, Theorem 2.3 implies that $K = 0$ is a locally stable fixed point for the controlled equation (1.11) if $c \in (c^*, 1)$ with

$$c^* = 1 - \frac{1}{\max\{|0|, |e^{1.5}| + 0\}} = 1 - \frac{1}{e^{1.5}} \approx 0.78.$$

Or in other words, that any constant harvesting effort greater than c^* is able to control the pest if the initial population is small enough.

In order to calculate the harvesting effort to control the pest independently of the initial population size, we note that condition (3.1) holds with $L = e^{1.5}$ and $K = 0$ because

$$|x \exp(1.5 - y)| \leq e^{1.5} |x| \leq e^{1.5} \max\{|x|, |y|\}, \quad (x, y) \in \mathbb{R}_+^2.$$

Thus, Corollary 3.2 implies that $K = 0$ is indeed a globally asymptotically stable fixed point for c greater than $1 - \frac{1}{e^{1.5}} \approx 0.78$. Figure 2 illustrates this example. Note how only for c greater than 0.78 the trivial fixed point is stabilized. Therefore, the estimates obtained from Theorem 2.3 and Corollary 3.2 cannot be improved in this case.

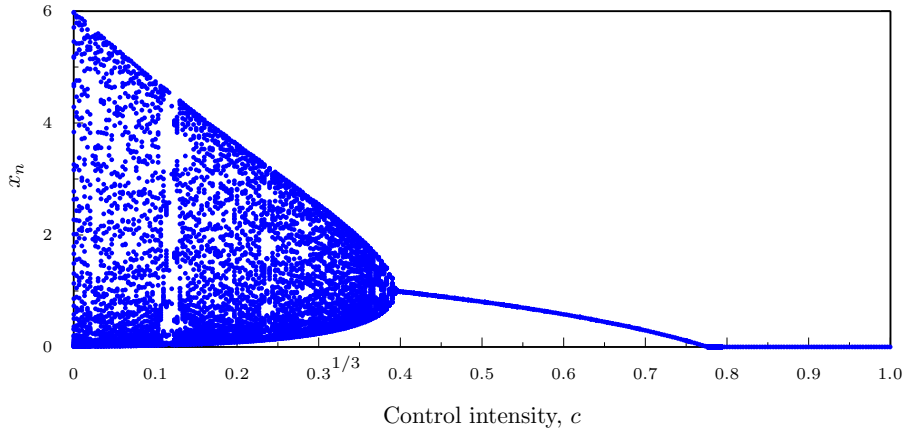


Figure 2: Stabilizing effect of increasing the control parameter c in the controlled (1.11). The uncontrolled equation is $u_{n+1} = u_n \exp(1.5 - u_{n-1})$. For each $c \in k/300$, $k = 1, \dots, 300$, we plotted 50 consecutive values of u_n after discarding the first 3000. Initial conditions were chosen pseudo-randomly.

5.3. Targeting. Stabilization of an arbitrary value.

The aim of this example is to show that using a combination of HMTOCs is very flexible from the point of view of targeting, that is, a suitable combination of HMTOCs allows us to carry the system into a desired objective. Let us consider the third order delayed Pielou equation

$$u_{n+1} = \frac{8u_n}{1 + u_{n-2}},$$

which was proposed as a discrete analogue of the logistic differential equation with delay [25].

To illustrate that a combination of two HMTOCs can stabilize any point $K \in \{\frac{8x}{1+x} : x > 0\} = [0, 8)$, let us choose, for example, $K = 6$, which is not a fixed point of the uncontrolled equation. It is easy to verify that taking $c_K = 2/9$ and $T_K = 3$ solves the equation

$$K = c_K T_K + (1 - c_K) \frac{8 \cdot K}{1 + K}.$$

Therefore, by Theorem 4.4 we know that point $K = 6$ is stabilized when using alternatively HMTOC with $c = c_K, T = T_K$ and with $T = K$ and $c \in (0, 1)$ if c is large enough in latter. Figure 3 numerically illustrates it.

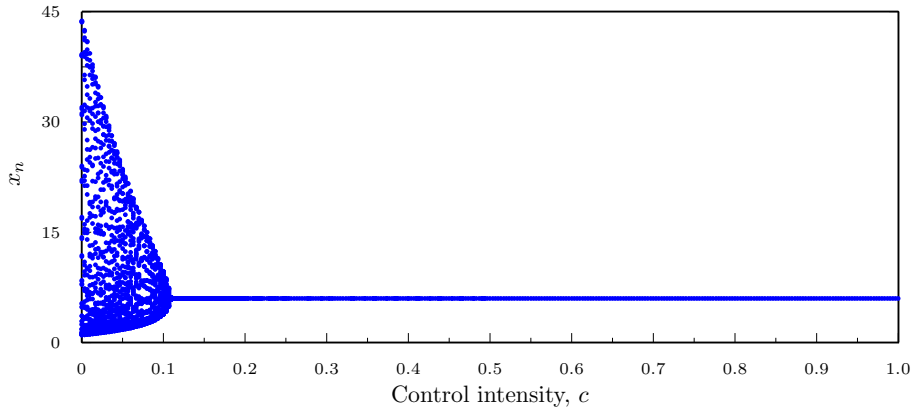


Figure 3: Stabilizing effect of a combination of two HMTOCs for the third order Pielou equation $u_{n+1} = \frac{8u_n}{1+u_{n-2}}$. The combination was chosen to guarantee the stabilization of $K = 6$. For each $c \in k/300, k = 1, \dots, 300$, we plotted 50 consecutive values of u_n after discarding the first 3000. Initial conditions were chosen pseudo-randomly.

6. Summary

Considering stabilization of higher order equations by a natural generalization of target oriented control [9], we have obtained:

1. sufficient local stabilization results;
2. sufficient global stabilization conditions;
3. estimates of the control intensity necessary to achieve stabilization.

In the second order case, the estimate of the minimum control intensity needed to locally stabilize a fixed point given in Theorem 2.3 is sharp as the first and second examples illustrate. Moreover, the second example shows that, at least in some cases, Corollary 3.2 is optimal. On the other hand, the numerical simulations suggest that our main global stability result (Theorem 3.1) could be improved. As remarked, one way to attain this improvement

will be to obtain the smallest constant L satisfying condition (3.1). Another one could be to study a restricted family of maps, as in [13] where the results hold for unimodal maps with a negative Schwarzian derivative. But probably this improvement will be difficult. This expected difficulty should not be surprising if we recall that the formulated in 1976 conjecture about local stability implying global stability for the delayed Ricker equation [20] was only recently solved for the second order case using a computer aided proof [2].

Most of stabilization results in the present paper were developed in the case when a fixed point of the original difference equation is stabilized. Later on, we describe how the target oriented control can shift a fixed point to any prescribed value. In this context, the scheme how all the previous stabilization results can be applied is the following:

- We define c_1 and T_1 so that the prescribed value x^* is a fixed point of the equation $g(\mathbf{x}) = c_1 T_1 + (1 - c_1)f(\mathbf{x})$.
- For the new higher order equation $\mathbf{x}_{n+1} = g(\mathbf{x}_n)$, the chosen x^* is a fixed point, so any of previous results on either local or global stabilization apply, with $T_2 = x^*$. In particular, these results allow to find bounds for $c_2 \in (c^*, 1)$ leading to stabilization. The process follows the proof of Theorem 4.4.
- By Lemma 4.1, we combine two successive HMTOCs getting $c_3 = c_1 + c_2(1 - c_1)$, where c_1 is fixed, $c_2 \in (c^*, 1)$, so $c_3 \in (c_1 + c^*(1 - c_1), 1)$, and $T_3 = (c_1 T_1 + c_2 x^*(1 - c_1))/c_3$ is uniquely defined once c_3 is chosen. Thus, we know the resulting c_3 and T_3 required for stabilization of x^* .

In population dynamics, when the population has age or spatial structure, this leads to systems of difference equations. In some relevant cases the system can be rewritten as a delayed equation (see, for example, [22], where a juvenile-adult population is considered) to which HMTOC could be applied. The method presented in this paper does not apply in the general system setting. However, the extension to systems should be straightforward from the results given here and will be discussed elsewhere.

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Appendix A.

Proof of Lemma 2.1. To prove claim (i) we note that fixed points of HMTOC are solutions of

$$x = g(x) := cT + (1 - c)f(x, \dots, x), \quad (\text{A.1})$$

where g is a continuous function. We have

$$g(0) = cT + (1 - c)f(0, \dots, 0) \geq cT > 0,$$

and

$$g(M) = cT + (1 - c)f(M, \dots, M) \leq cT + (1 - c)M \leq M.$$

Thus, equation (A.1) has at least one positive solution $x \in (0, T)$ by the intermediate value theorem.

Claim (ii) follows from claim (i) by taking $M = \max\{T, \overline{M}\}$. \square

Proof of Theorem 2.2. It is well known that a fixed point $P_c \in I$ of HMTOC is asymptotically stable if $\{z \in \mathbb{C} : |z| < 1\}$ contains all the roots of the characteristic polynomial

$$p_{P_c}(x) := x^k - \frac{\partial f}{\partial x_1}(\mathbf{P}_c) x^{k-1} - \dots - \frac{\partial f}{\partial x_k}(\mathbf{P}_c)$$

where $\mathbf{P}_c = (P_c, \dots, P_c)$; see [7, Section 1.2] for more details.

Applying Fujiwara's upper bound (see for instance [10, Remark 8]) to polynomial p_{P_c} , we obtain that any root λ of p_{P_c} satisfies

$$|\lambda| \leq 2(1 - c) \max_{i=1, \dots, k} \left| \frac{\partial f}{\partial x_i}(\mathbf{P}_c) \right|^{\frac{1}{i}}.$$

Next, since the partial derivatives of f are bounded on the rectangle I^k , there exists a positive real number A such that

$$A = \max_{P_c \in I} \left\{ \max_{i=1, \dots, k} \left| \frac{\partial f}{\partial x_i}(\mathbf{P}_c) \right|^{\frac{1}{i}} \right\}. \quad (\text{A.2})$$

Therefore, taking

$$c^* = \max \left\{ 0, 1 - \frac{1}{2A} \right\},$$

we obtain that all the roots λ of the polynomial p_{P_c} satisfy $|\lambda| < 1$ for $c \in (c^*, 1)$. \square

Proof of Theorem 2.3. Following the reasoning of Theorem 2.2, we have to guarantee that all the roots of the characteristic polynomial

$$x^2 - (1 - c) \frac{\partial f}{\partial x}(\mathbf{K})x - (1 - c) \frac{\partial f}{\partial y}(\mathbf{K}),$$

are smaller than one in modulus. Applying the Jury conditions [17, 19], we obtain that a necessary and sufficient condition for that to happen is

$$2 > 1 - (1 - c) \frac{\partial f}{\partial y}(\mathbf{K}) > (1 - c) \left| \frac{\partial f}{\partial x}(\mathbf{K}) \right|,$$

which is equivalent to

$$\left| \frac{\partial f}{\partial y}(\mathbf{K}) \right| < \frac{1}{1-c} \quad \text{and} \quad \left| \frac{\partial f}{\partial x}(\mathbf{K}) \right| + \frac{\partial f}{\partial y}(\mathbf{K}) < \frac{1}{1-c},$$

from which we obtain the desired result. \square

Proof of Theorem 2.4. The result follows from [7, Theorem 1.2.5], after noting that the conditions of the theorem imply that

$$\sum_{j=1}^k (1-c) \left| \frac{\partial f}{\partial x_j}(\mathbf{K}) \right| < 1.$$

\square

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